Use of Shear Lag for Composite Microstress Analysis—Rectangular Array

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A number of authors have employed shear lag theory to find the stress distribution near cracks in a uniaxially reinforced composite. Practical application of these equations has been hampered by the fact that they include a parameter *h/d*, the magnitude of which is unknown. This parameter is evaluated for composites in which the fibers form a rectangular array.

Nomenclature

d	= distance between rivet lines or distance between rod or fiber centers
d_1	= distance between next-to-nearest fiber pair
d_2	= distance between nearest fiber pair
F_i	= force in the ith stiffener or fiber in a composite
G	= shear modulus of the matrix
Gh	= effective shear stiffness of the composite
h	= plate or shell thickness
$(h/d)_{c,h}$	= h/d of the composite in the horizontal direction
$(h/d)_{c,v}$	= h/d of the composite in the vertical direction
I	= current generated by the power supply
I_i	= current in the ith rod
j	= current density in electrolyte
j_n	= normal component of the current density
ℓ	= submerged length of the rods
r_0	= radius of a fiber or a rod
r_1	= distance from a negative line force
r_2	= distance from a positive line force
w_n	= axial displacement of the nth fiber
z	= axial coordinate
α	= d_1/d_2
ρ_e	= resistivity of the electrolyte
τ	= shear stress in matrix
ϕ	= electrical potential in rod or electrolyte

Introduction

THE analysis of failure of initially crack-free continuous-fiber composites is complicated because such failures are not governed by weakest-link considerations. Instead they are a result of damage accumulation. To account for crack growth, we must know the microstresses in the neighborhood of the crack tip. At present such understanding is limited to unidirectionally reinforced composites and, even there, is somewhat fragmentary.

Most attempts to find stress distributions near the crack tip are based on shear lag theory. This approach simplifies elastic theory by assuming that all displacements are parallel to the fibers and that the matrix between the fibers carries only shear stress. The fundamental equation of shear lag theory, originally developed for the stiffened plates and shells employed in aircraft, takes the form

$$\frac{\mathrm{d}F_i}{\mathrm{d}z} = Gh\left(\frac{-w_{i-1}+2w_i-w_{i+1}}{d}\right) \tag{1}$$

By analogy, the corresponding equations for composites were assumed by Hedgepeth[1,2] and later by many others to take the same form in the case of a linear array, and in the case of a square array of fibers to take the form

$$\frac{\mathrm{d}F_{ij}}{\mathrm{d}z} = F'_{ij} = Gh\left(\frac{-w_{i,j-1}-w_{i-1,j}+4w_{i,j}-w_{i,j+1}-w_{i+1,j}}{d}\right) \tag{2}$$

This would be correct for a square array if it had a very low fiber volume ratio and the matrix was replaced by thin plates as shown in Fig. 1. But for high fiber volume ratio and/or conventional matrices, it is quite unclear how to estimate h/d or whether the assumption that only nearest neighbors interact is a good one.

Some work has already been done on these questions. Batdorf[3] gave an exact solution for the interaction between two infinitely long rigid rods immersed in an infinite elastic matrix with displacements $\pm w_0$. The result in this case was that

$$\frac{\mathrm{d}F_+}{\mathrm{d}z} = -\frac{\mathrm{d}F_-}{\mathrm{d}z} = G\frac{h}{d}\cdot 2w_0 \tag{3}$$

where

$$\frac{h}{d} = \pi/\ell n\left[\left(\frac{1\pm\sqrt{1-(2r_0/d)^2}}{\{2r_0/d\}}\right)\right] \tag{4}$$

Here r_0 is the rod radius and d the distance between rod centers. It follows that h/d is a function of r_0/d varying from 0 for $r_0=0$ to ∞ for $r_0=d/2$. Unfortunately the technique employed cannot be extended to more than two fibers. This is because, whereas the displacement contours in the matrix surrounding two rods of circular cross section are circular cylinders, for more than two rods they are not.

The interaction between nearest neighbors for the case of a square array was found by Batdorf and Ghaffarian[4] using an electric analog.[5] The present paper extends this approach to the determination of interaction between nearest neighbors in a rectangular array.

The interaction for rectangular arrays is of interest for several reasons. One is that if a prepreg is compressed in the thickness direction during cure, the mean fiber separation in the thickness direction may be less than in the in-plane direction. Another is that the theory for a rectangular array is useful in analyzing a linear array for arbitrary matrix thickness. In addition, variable spacing gives a measure of flexibility in composite design.

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To supplement the solution found experimentally employing the electric analog, an analytical approach valid for very small fiber volume ratio is presented. The two solutions are found to be in good agreement in the limiting case where the analytical solution is valid.

Analytical Solution for Rectangular Array

For the case of a rectangular array, two parameters have to be taken into account at the same time; namely, h/d for the two nearest neighbors and h/d for the next-to-nearest neighbors. The analysis for such an array can be carried out by a consideration of the load transfer due to a set of equal but opposite forces directed to two nearest neighbors, as in Fig. 2, and to two next-to-nearest neighbors, as in Fig. 3. The analysis is valid in the limiting case of small r_0/d.

In case A, the shear lag equation becomes

$$F_a' = G\left[\left(\frac{h}{d}\right)_{c,v}(2w_a - w_c - w_e) + \left(\frac{h}{d}\right)_{c,h}(2w_a - w_b - w_d)\right] \quad (5a)$$

$$= G\left(\frac{h}{d}\right)_{i,v}(w_a - w_c) \quad (5b)$$

where subscript c denotes the value for the two-fiber interaction in a rectangular array composite, and i denotes the value for a pair of isolated fibers. For sufficiently small r_0/d_2, it has been shown that

$$w_a = -w_c = \frac{F'}{2\pi G}\ln\left(\frac{d_2}{r_0} - 1\right) \quad (6)$$

The displacements of the other fibers can be obtained via

$$w_i = \frac{F'}{2\pi G}\ln\left(\frac{r_1}{r_2}\right), \quad i \neq a \text{ or } c \quad (7)$$

where the field point is the center of the fiber whose displacement is being calculated. Thus

$$w_b = \frac{F'}{2\pi G}\ln\sqrt{1 + 1/\alpha^2}$$

$$= w_a \frac{\ln\sqrt{1+1/\alpha^2}}{\ln(d_2/r_0 - 1)} \quad (8)$$

$$w_d = w_b \text{ (by symmetry)} \quad (9)$$

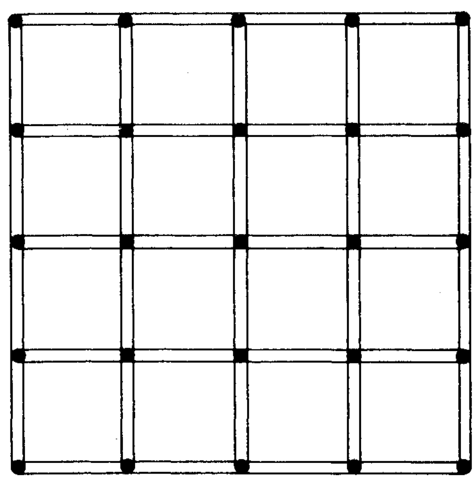

Fig. 1 Hypothetical material obeying conventional shear lag equations.

and

$$w_e = w_a \frac{\ln 2}{\ln(d_2/r_0 - 1)} \quad (10)$$

Thus, Eq. (5a) can be written as

$$F_a' = G\left[\left(\frac{h}{d}\right)_{c,v}\left(1 + 1/2 - \frac{\ln\sqrt{2}}{\ln(d_2/r_0 - 1)}\right) + \left(\frac{h}{d}\right)_{c,h}\left(1 - \frac{\ln\sqrt{1+1/\alpha^2}}{\ln(d_2/r_0 - 1)}\right)\right](2w_a) \quad (11a)$$

$$= G\left(\frac{h}{d}\right)_{i,v}(w_a - w_c) \quad (11b)$$

Employing Eq. (4) for h/d for a pair of isolated fibers, $(h/d)_{i,v}$ can be replaced by

$$\frac{\pi}{\ln[(1 \pm \sqrt{1-(2r_0/d_2)^2})/(2r_0/d_2)]} \quad (12)$$

Again by noting that $w_a = -w_c$, the above equality becomes

$$\left(\frac{h}{d}\right)_{c,v}\left(1.5 - \frac{\ln\sqrt{2}}{\ln(d_2/r_0 - 1)}\right) + \left(\frac{h}{d}\right)_{c,h}\left(1 - \frac{\ln\sqrt{1+1/\alpha^2}}{\ln(d_2/r_0 - 1)}\right) = \frac{\pi}{\ln[(1 \pm \sqrt{1-(2r_0/d_2)^2})/(2r_0/d_2)]} \quad (13)$$

The above equation is a linear algebraic equation in two unknowns $(h/d)_{c,v}$ and $(h/d)_{c,h}$. A second equation can be found by applying a similar approach to case B. For this case we obtain

$$\left(\frac{h}{d}\right)_{c,v}\left(1 - \frac{\ln\sqrt{1+\alpha^2}}{\ln(d_1/r_0 - 1)}\right) + \left(\frac{h}{d}\right)_{c,h}\left(1.5 - \frac{\ln\sqrt{2}}{\ln(d_1/r_0 - 1)}\right) = \frac{\pi}{\ln[(1 \pm \sqrt{1-(2r_0/d_1)^2})/(2r_0/d_1)]} \quad (14)$$

Solving Eqs. (13) and (14) simultaneously, we obtain

$$\left(\frac{h}{d}\right)_{c,h} = \frac{A - (B/F)C}{D - C(E/F)} \quad (15)$$

and

$$\left(\frac{h}{d}\right)_{c,v} = \frac{B - (h/d)_{c,h}(E)}{F} \quad (16)$$

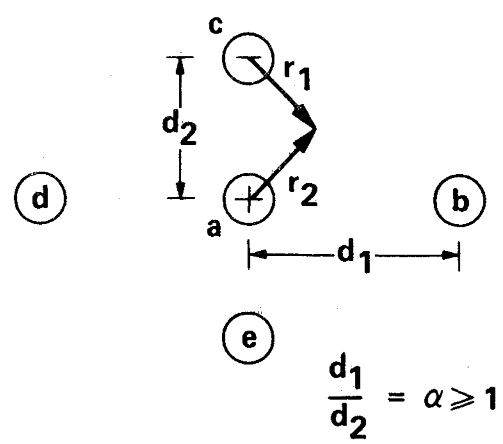

Fig. 2 Loads applied to nearest neighbor pair—case A.

where

$$A=\frac{\pi}{\ln[(1\pm\sqrt{1-(2r_0/d_1)^2})/(2r_0/d_1)]}$$

$$B=\frac{\pi}{\ln[(1\pm\sqrt{1-(2\alpha r_0/d_1)^2})/(2\alpha r_0/d_1)]}$$

$$C=1-\frac{\ln\sqrt{1+\alpha^2}}{\ln(d_1/r_0-1)}$$

$$D=1.5-\frac{\ln\sqrt{2}}{\ln(d_1/r_0-1)}$$

$$E=1-\frac{\ln\sqrt{1+1/\alpha^2}}{\ln(d_1/\alpha r_0-1)}$$

$$F=1.5-\frac{\ln\sqrt{2}}{\ln(d_1/\alpha r_0-1)}$$

Theory Underlying Electric Analog

If a set of n infinitely long rigid rods immersed in an infinite elastic matrix are displaced axially from their equilibrium positions by distances $w_1, w_2, \ldots w_n$, the distorted shape of every cross section will be the same. Therefore there will be no direct stress in the matrix, and the only shear stresses will be $\tau_{xz}=\tau_{zx}$ and $\tau_{yz}=\tau_{zy}$. At the interface between the ith rod and the matrix, the equilibrium equation is

$$\frac{dF_i}{dz}=\oint_i(\tau_{zx}dy+\tau_{zy}dx) \tag{17}$$

where $\oint$ signifies integration around the circumference of the ith rod. In the matrix, equilibrium in the z direction requires that

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}=0 \tag{18}$$

Since the only displacement is $w(x,y)$, application of Hooke's law leads to

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0 \tag{19}$$

in the matrix, and

$$\frac{dF_i}{dz}=G\oint_i\left(\frac{\partial w}{\partial y}dx+\frac{\partial w}{\partial x}dy\right) \tag{20}$$

at the interface with the ith fiber.

Next, consider a geometrically similar array of n perfectly conducting rods immersed in an electrolyte of resistivity ρ_e. Let the potentials applied to the rods be $\phi_1, \phi_2 \ldots \phi_n$. The current flowing from the electrolyte into a unit length of the ith rod is then

$$\oint_i(j_y dx+j_x dy)=\frac{dI_i}{dz} \tag{21}$$

In the electrolyte, Ohm's law leads to

$$\rho_e j=\text{grad}\,\phi \tag{22}$$

and for steady currents

$$\nabla\cdot j=0 \tag{23}$$

Thus, in the electrolyte,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{24}$$

and at the periphery of a rod,

$$\frac{dI_i}{dz}=\frac{1}{\rho_e}\oint_i\left(\frac{\partial\phi}{\partial y}dx+\frac{\partial\phi}{\partial x}dy\right) \tag{25}$$

Comparing Eqs. (19) and (20) with Eqs. (24) and (25), we see that ϕ is analogous to w, I to F, and ρ_e to $1/G$.

The analogy provides a convenient means for an experimental determination of the interaction between fibers. If we assume the validity of Eq. (2), we can write

$$\frac{dI_{ij}}{dz}=\frac{h}{\rho_e d}[-\phi_{i,j-1}-\phi_{i-1,j}+4\phi_{i,j}-\phi_{i+1,j}-\phi_{i,j+1}] \tag{26}$$

If we measure the total current per unit length in the ith rod when known potentials are applied to that rod and its four nearest neighbors, we can use Eq. (26) to determine $h/\rho_e d$. By measuring ρ_e we can find h/d for a square array. This was done in Ref. 4.

The case of a rectangular array is a bit more complicated. The square array leads to Eq. (26), a linear algebraic equation in one unknown, h/d. As noted in the preceding section, in a rectangular array there are two unknowns, h/d for the two nearest neighbors and h/d for the next-to-nearest neighbors. Thus two experiments must be carried out, leading to two simultaneous equations in two unknowns.

Experimental Setup

The structure of the composite was simulated in the electric analog by conducting rods replacing the fibers and a weak electrolyte replacing the matrix. A transparent plastic tank was used to contain the fluid and the rods.

The experiment was carried out first by inserting one end of the rods into a plexiglass panel with holes that match the size of the rods and with spacing that meets the r_0/d and α under investigation. This setup was then loaded into the electrolyte. Alignment of the electrodes was provided by inserting the other ends into a similar panel that can be mounted at any convenient level above the electrolyte.

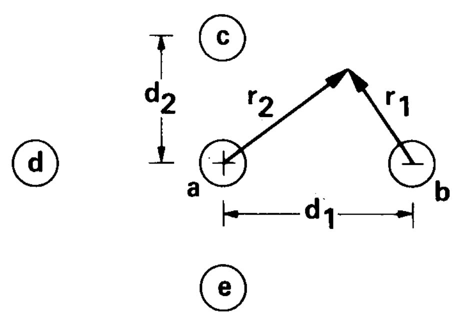

Fig. 3 Loads applied to next-to-nearest neighbor pair—case B.

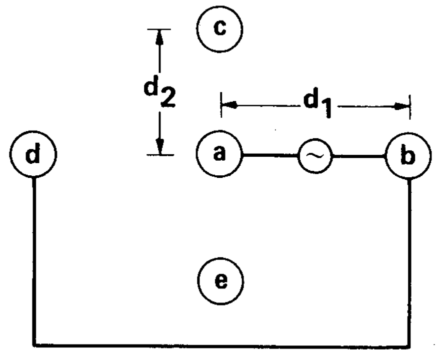

Fig. 4 Potentials applied to next-to-nearest neighbor pair.

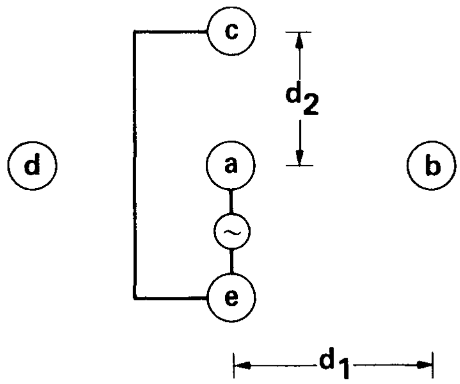

Fig. 5 Potentials applied to nearest neighbor pair.

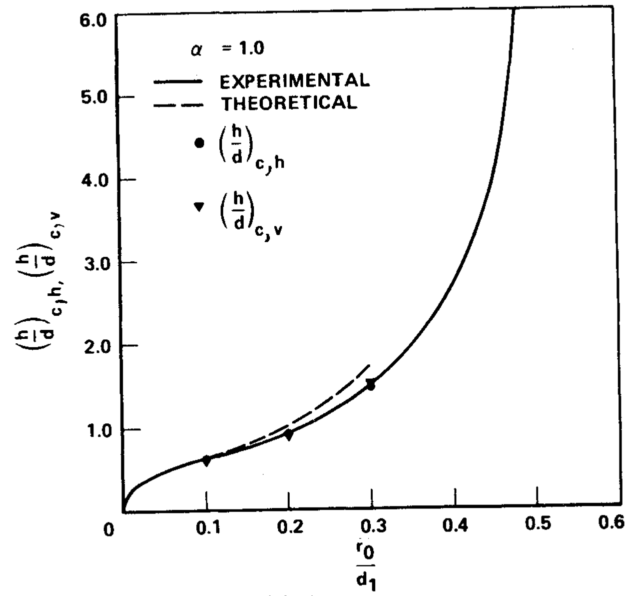

Fig. 6 h/d for square array.

Two experiments were performed for each α and each r_0/d. The first one involved the application of a constant ac potential across rods a and b as in Fig. 4. The two outermost rods were connected, and thus d was at the same potential as b, while c and e were left free to adopt the local potential of the electrolyte.

The governing equation can be written as

$$\frac{dI}{dz}=\frac{I(0)}{\ell}=\frac{1}{\rho_e}\left[\left(\frac{h}{d}\right)_{c,v}(\phi_{ac}+\phi_{ae})+\left(\frac{h}{d}\right)_{c,h}(\phi_{ab}+\phi_{ad})\right] \tag{27}$$

$I(0)$ was obtained by measuring the current generated by the power supply; ℓ is the submerged length of the rods. ϕ_{ac} and ϕ_{ab} were obtained by measuring the potential difference between a and c, and a and b, respectively. By symmetry, $\phi_{ae}=\phi_{ac}$.

In order to obtain a second equation relating the two unknowns, $(h/d)_{c,v}$ and $(h/d)_{c,h}$, a constant potential was next applied across rods a and e with e connected to c as in Fig. 5. The preceding governing equation again applies. Similarly, I, ϕ_{ae}, and ϕ_{ab} were measured for this case. By solving the two linear algebraic equations simultaneously, values for $(h/d)_{c,v}$ and $(h/d)_{c,h}$ were obtained.

Results

The results are summarized in Figs. 6-11. Figure 6 gives h/d as a function of r_0/d_l for a square array, i.e., $\alpha=1$. This case was treated earlier in Ref. 4. The other figures treat the cases

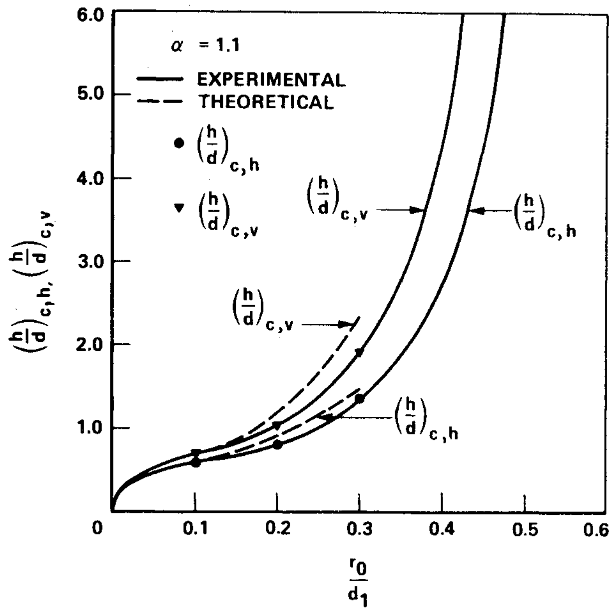

Fig. 7 h/d for rectangular array.

$\alpha=1.1$ to $\alpha=1.5$. For $\alpha>1$, the array is rectangular and therefore there are two curves, one representing (h/d) for the vertical interaction and the other for the horizontal interaction. In each case, d in the abscissa was chosen to be the longer distance d_l; this was of course an arbitrary decision. In the case of the ordinates, d is d_l for the horizontal interaction and d_2 for the vertical interaction.

In every case h/d approaches zero as r_0/d_l approaches zero, and in this region the theoretical solution is valid. It is therefore used to estimate the value of h/d at such small values of r_0/d_l as to make an experimental determination difficult.

As $r_0/d_l\rightarrow 0.5$, the horizontal separation between two fibers vanishes and $(h/d)_{c,h}\rightarrow\infty$. This asymptote can be used to extrapolate the experimental data to higher values of r_0/d_l than can readily be investigated experimentally. As $r_0/d_2\rightarrow 0.5$, the value of $(h/d)_{c,v}\rightarrow\infty$ because then the vertical separation shrinks to zero. In this case $r_0/d_l=r_0/\alpha d_2=0.5/\alpha$, which serves similarly as an asymptote.

Discussion

In Hedgepeth's papers,[1,2] no attempt was made to evaluate h/d for a particular composite, and no technique was proposed for doing so. These papers evaluated the stress concentration factors in fibers adjacent to a crack, and it turns out that the stress concentration factor does not depend on the value of h/d. However, to find the stresses at which crack extension occurs, it is necessary to find lengths of the overloaded fiber elements, and these do depend on h/d. In the absence of a theoretical technique for accomplishing this, various assumptions have been made. The most common has been to assume that for the vertical interaction (interaction between closest neighbors) $d=d_2$ and $h=d_l$, while for the horizontal interaction $d=d_l$ and $h=d_2$.[6,7] These assumptions lead to the results

$$(h/d)_{c,v}=\alpha \tag{28a}$$

$$(h/d)_{c,h}=1/\alpha \tag{28b}$$

a result independent of r_0/d. Comparing these results with Figs. 6-11, we see that they are somewhere near right for $r_0/d_l=0.2$ but are too high for smaller values of r_0/d_l and too low for higher values of r_0/d_l. We note that when r_0/d_l is 0.2, the fiber volume ratio is about $\alpha/8$, a lower value than what is usually encountered in practical applications. For a square array having a fiber volume ratio 0.5

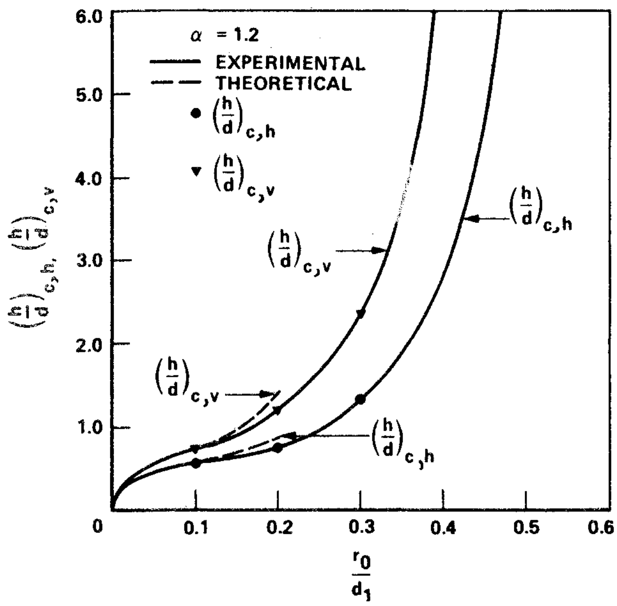

Fig. 8 *h/d* for rectangular array.

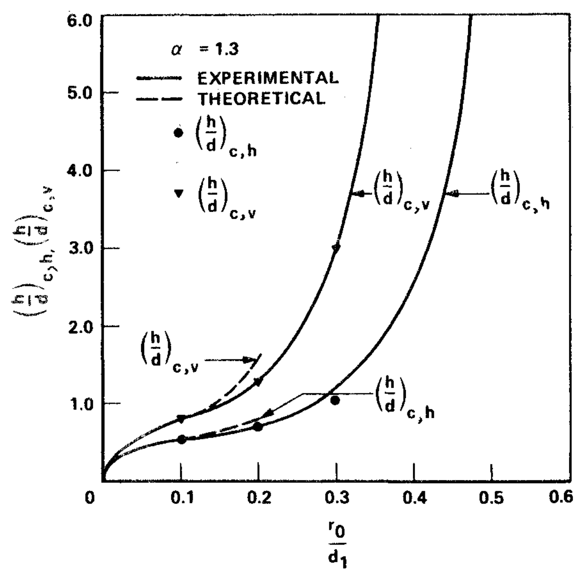

Fig. 9 *h/d* for rectangular array.

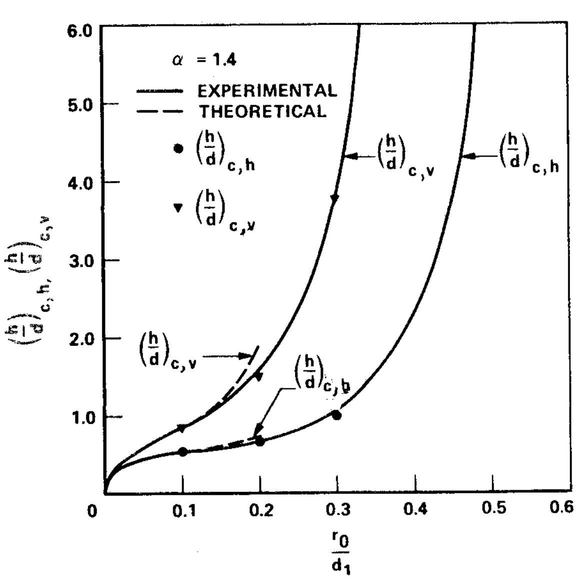

Fig. 10 *h/d* for rectangular array.

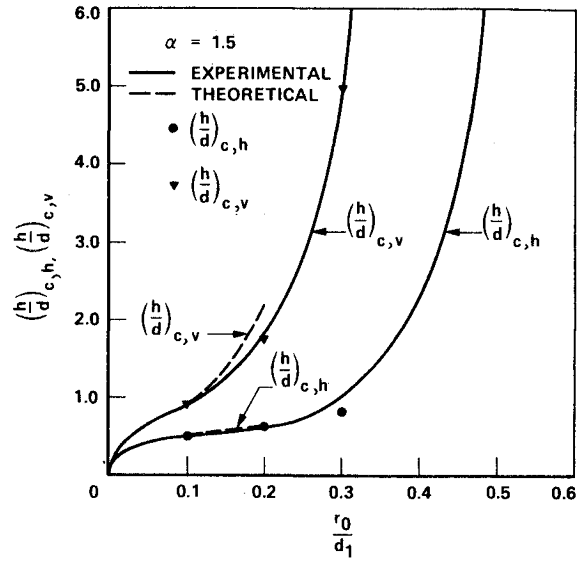

Fig. 11 *h/d* for rectangular array.

$r_0/d \approx 0.4$, in which case the value we obtain (see Fig. 6) is about 2.5 times that given by Eq. (28).

Concluding Remarks

During the last three decades, shear lag has been widely used as a technique for finding the stresses in damaged uniaxially reinforced composites. The applicable equations contain a parameter representing the shear interaction between neighboring fibers. Somewhat surprisingly, up to now little has been known about how to evaluate this term for a particular composite.

It is shown that for small values of the fiber diameter to fiber separation, this parameter can be evaluated theoretically. For larger values it can be evaluated experimentally employing an electric analog. By combining the two techniques, the parameter is evaluated over the entire range of diameter-to-separation ratio for composites in which the fibers form a rectangular array. The results contained in Figs. 6-11 cover arrays ranging from square to rectangles having a side ratio of 1.5.

Acknowledgments

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